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H.W. 1
Ch. 2: Groups

Q.4] $GL(2,R)$, non-abelian :-

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $|A| = 2 \neq 0$

and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $|B| = 1 \neq 0$

$AB \neq BA$

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, non-abelain

Q.6] $a^{-1}ba \neq b$:

$G(Z_2, \oplus_2)$

	0	1
0	0	1
1	1	0

let $a = 0 \rightarrow a^{-1} = 0$
and let $b = 1$
$a^{-1}ba = (0)(1)(0) = 0 \neq 1 \neq b$.

Q.11] ① closed, $\det(AB) = (\det A)(\det B) = (1)(1) = 1 \Rightarrow AB \in G$

② Identity, yes, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since $|e| = 1$

③ Inverse :- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ its determant is
$ad - cb = 1$ $\qquad ad - bc = 1$

④ Matric multiplication is associative
$A, B, C \in G$
$(A.B)C = A.(B.C)$
$\longrightarrow$ is a group matrix.

Q.12] any integer $n > 2$;-

$U(n) = \{1, \dots, n-1\}$

$1^2 = 1$, $(n-1)^2 = 1$ → $(n-1)^2 = n^2 - 2n + 1$

ex. $U(9) = \{1, \dots, 8\}$ $= (n-2)n + 1$

$1^2 = 1$, $8^2 = 1$ $= 1$, since $n = 0$ always.

Q.14] G is group with the following property whenever a, b and c belong to G and $ab = ca$ then $b = c$, prove G abelian:-

$\forall a, b \in G$

$\cancel{a}ba = ab\cancel{a}$ (cancel from left and right).

$ba = ab$

Q.17] G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$:-

$\boxed{\Rightarrow}$ suppose that G is abelian

$\Rightarrow (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$

$\boxed{\Leftarrow}$ if $(ab)^{-1} = a^{-1}b^{-1}$

$\Rightarrow (ab)(a^{-1}b^{-1}) = e$ (multiplying both sides on the right by ba).

$\Rightarrow ba = ab$

$\rightsquigarrow$ abelian

$(ab)(a^{-1}b^{-1})(ba) = ba$

$ab(a^{-1}a) = ba$

Q.18] $(a^{-1})^{-1} = a$ for all a :-

$aa^{-1} = e$ (multiplying from right by $(a^{-1})^{-1}$)

$aa^{-1}(a^{-1})^{-1} = e(a^{-1})^{-1}$

$a(a^{-1}(a^{-1})^{-1}) = (a^{-1})^{-1}$

$ae = (a^{-1})^{-1}$

$\Rightarrow a = (a^{-1})^{-1}$ #

Q.20] If $a_1, a_2, \dots, a_n$ belong to a group what is the inverse of $a_1 a_2 \dots a_n$?

$\Rightarrow (a_1 a_2 \dots a_{n-1} a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1}$

$\Rightarrow (a_1 a_2 \dots a_{n-1} a_n)(a_n^{-1} a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1}) = e.$

Q.24] $U(12) = \{1, 5, 7, 11\}$

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Q.25]

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Q.26] if $(ab)^2 = a^2b^2$ in a group G then $ab = ba$:-

$\Rightarrow (ab)^2 = a^2b^2$

$\cancel{a}ba\cancel{b} = \cancel{a}ab\cancel{b}$

$\Rightarrow ba = ab$

Q.28] ① closure: $(3^m 6^n)(3^s 6^t) = 3^{m+s} 6^{n+t} \in G$

② associativity ✓ $(3^{m_1} 6^{n_1} . 3^{m_2} 6^{n_2}) . 3^{m_3} 6^{n_3} = 3^{m_1} 6^{n_1} . (3^{m_2} 6^{n_2} . 3^{m_3} 6^{n_3})$

$3^{m_1+m_2+m_3} 6^{n_1+n_2+n_3} = 3^{m_1+m_2+m_3} 6^{n_1+n_2+n_3}$ ✓

③ identity: $3^0 6^0 = 1 = e$

④ Inverse: $(3^m 6^n)^{-1} = 3^{-m} 6^{-n}$, $3^m 6^n . 3^{-m} 6^{-n} = 1 = e$

$\Rightarrow$ group.

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Q.35] G is group with property that the square of every element is the identity, then G is abelian :-

$\Rightarrow$ since $a^2 = b^2 = (ab)^2 = e$, $a, b \in G$ closure.

$\Rightarrow$ $\cancel{a}abb\cancel{b} = \cancel{a}ba\cancel{b}$

$\Rightarrow$ $ab = ba$

$\Rightarrow$ abelian

Q.39] $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$:

$\Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$

and $2ab \neq 0$ $\rightarrow$ closure : ~ (1)

$\Rightarrow$ matrix multiplication is associative. ~ (2)

$\Rightarrow$ the Identity is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ ~ (3)

$\Rightarrow$ the inverse $\begin{bmatrix} a & a \\ a & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{bmatrix}$ ~ (4)

$\Rightarrow$ the group $GL(2,\mathbb{R})$ has different identity than the group G.